

## Orthogonal elementary interactions for bimatrix games

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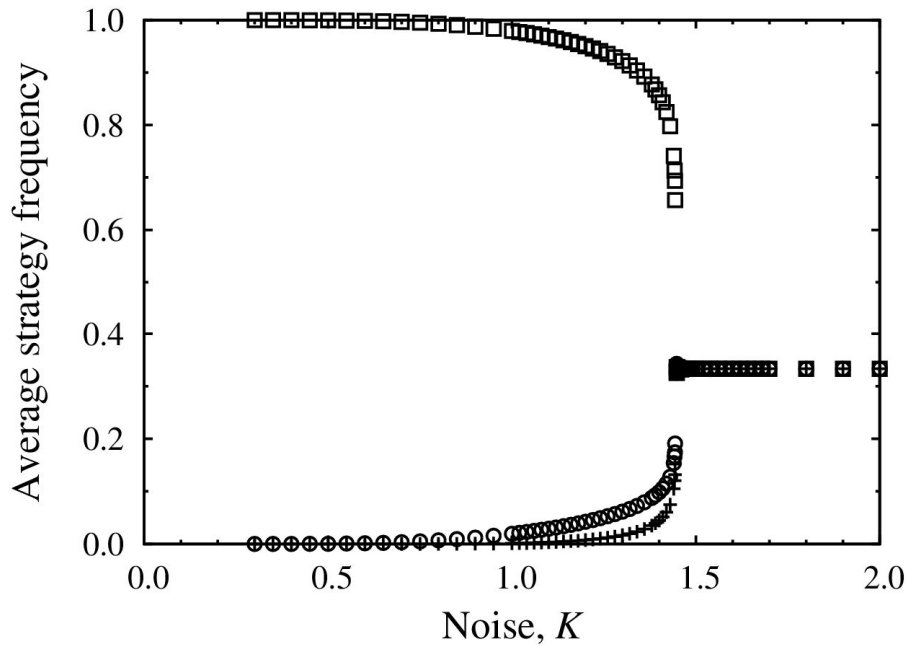
In evolutionary games a payoff matrix defines the interactions between two equivalent players if they can both choose one of  $n$  options (called strategies). More precisely, the elements of the payoff matrix quantify the incomes for all possible strategy pairings. Similarly to vectors, however, the matrices can also be built up as a linear combination of orthogonal basis matrices. For Cartesian-type basis matrices (containing a single 1 while all other matrix elements are 0) the coefficients define the incomes for all strategy pairings (identified by the location of the 1 in the matrix). Instead of this set of basis matrices, however, we can use another set of basis matrices that allows us to identify four types of orthogonal interactions in addition to the all-one matrix, which can be considered as an irrelevant term. For example, in games with self-dependent payoffs the matrices contain identical elements in the rows. For games with cross-dependent payoffs there are identical elements in the columns. After removing these components from a general game, the rest of the payoff matrix can be separated into the sum of a symmetric and anti-symmetric matrix. The symmetric part sums coordination-type interactions that measure the preference of identical (versus differing) strategy choices for all strategy pairs. On the other hand, the anti-symmetric part can be built up from a suitable subset of rock-paper-scissors type games.

These four types of orthogonal elementary interactions exhibit different characteristics and yield fundamental differences in the macroscopic behavior of multi-agent evolutionary games. There is no real interaction between the players if the game is defined by the symmetric combination of games with self- and cross-dependent payoffs. For these interactions the optimal strategy choice is dictated by external constraints for the selfish players. At the same time, the anti-symmetric combination of such games can generate a conflict (social dilemma) between individual and group interests. The presence of rock-paper-scissors type components prevents the existence of a potential and the possibility of thermodynamic behavior for otherwise suitable dynamical rules. At the same time, the latter zero-sum components give rise to a wide range of the macroscopic behaviors characterizing ecological and social systems.

In the present work the authors extend the classification of orthogonal elementary interactions to bimatrix games, in which two different players should choose one of their  $n$  options. In these game theoretical interactions the number of payoff parameters (dimension) is doubled. The application of the above-mentioned concept of matrix decomposition indicates the existence of all-one components and parts with self- and cross-dependent payoffs with independent parameters for the players. The rest of the bimatrix can be separated into the sum of anti-symmetric (zero-sum games) and symmetric (fraternal or partnership games) parts. It turns out that the zero-sum components can be built up from a suitable subset of matching pennies games, in which the players are allowed to use only two of their options [called “head” (H) and “tail” (T)]. The presence of these elementary interactions prevents the existence of a potential and also the possibility of thermodynamic behaviors, because it supports cyclic unilateral strategy changes (e.g., HH to HT to TT to TH to HH ...), which maintain important processes in living systems.

Another new and interesting interaction is found among the fraternal coordination-type interactions. These so-called “directed anti-coordination components” quantify the different

capabilities of players if  $n > 2$ , and involve features characteristic to coordination and rock-paper-scissors type games. More precisely, this interaction suggests the players to choose different strategies in specific configurations from which a strategy exchange would be disadvantageous for both. Monte Carlo simulations were performed to explore the general consequences of an elementary three-strategy directed anti-coordination interaction when the players are located on a square lattice. In this evolutionary game the participants play games with their nearest neighbors and can modify their strategy unilaterally to increase their own income. The numerical analyses of this model indicated that the system undergoes a continuous order-disorder phase transition as the noise level ( $K$ ) in the dynamical rule is tuned.



This figure illustrates that three different strategy frequencies occur in the ordered phase. Despite this difference, the critical transition reproduces the relevant robust features characteristic to the three-state Potts model.

In the light of the present results the concepts of orthogonal elementary interactions may be exploited for the systematic exploration of complex phenomena when considering the interplay between several orthogonal components.

Reference:

<https://www.sciencedirect.com/science/article/pii/S0378437125005722>